MATHEMATICS

ON TORSION-FREE COTORSION GROUPS

BY

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Introduction

In this paper we want to investigate the structure of the torsion-free cotorsion groups, defined by HARRISON [3], p. 372. For convenience, let us summarize some results of [3] that will be applied here. The word group will always mean abelian group. The additive group of rationals is denoted by Q, the additive group of integers is denoted by Z. If G is a group, then G_t is the torsion subgroup of G. A group is reduced if it has no nontrivial divisible subgroup. A reduced group G is called *cotorsion* if G a subgroup of a group M with M/G torsion-free imply that G is a direct summand of M, i.e. Ext $(H, G)=0$ for all torsion-free groups H. A cotorsion group is called *adjusted* if it has no torsion-free direct summands. The main result of HARRISON [3] states:

(i) Every cotorsion group is uniquely the direct sum of a torsion-free cotorsion group and an adjusted cotorsion group. A cotorsion group is adjusted if and only if G/G_t is divisible.

A result of FUCHS [2], p. 123 states:

(ii) A torsion-free group is a cotorsion group if and only if it is a reduced algebraically compact group.

Here an algebraically compact group is defined as a group which is a direct summand in every group that contains it as a pure subgroup. Our result on the structure of torsion-free cotorsion groups may be applied to the torsion-free reduced algebraically compact groups. In [3], Prop. 2.1., p. 371 it is proved that

(iii) A group is torsion-free cotorsion if and only if it is isomorphic to a direct summand of a direct product (unrestricted direct sum) of p-adic integers. E. WALKER [6] has investigated direct summands of direct products of abelian groups. He obtains that a direct product G of cotorsion groups has a maximal torsion-free direct summand H, and $H=0$ if and only if G/G_t is divisible. Furthermore, any non-zero torsion-free direct summand of G is uncountable.

By our methods we can describe the structure of such a maximal torsion-free direct summand H, since H, as a direct summand of G, is again cotorsion (Theorem 1). An alternative for the condition "G/G_t is divisible" is given in Theorem 2, which is easier to handle in special cases.

Finally, we discuss the structure of $\sum_{i=1}^{\infty}{}^{*} G_i / \sum_{i=1}^{\infty} G_i$, where G_i is a cyclic group of order p^i for some fixed prime p. It is known that the maximal torsion subgroup of $\sum_{i=1}^{\infty}{}^{*} G_i$ is not a direct summand of $\sum_{i=1}^{\infty}{}^{*} G_i$ [4]. However, $\sum_{i=1}^{\infty}{}^{*} G_i$ does have a maximal torsion-free direct summand $H \neq 0$ and H, an interdirect sum of groups of p-adic integers, appears also as a direct summand in the direct decomposition of the group $\sum_{i=1}^{\infty}{}^{*} G_i / \sum_{i=1}^{\infty} G_i$.

I would like to thank Prof. L. Fuchs for his help in many questions while I was preparing this paper.

1. *The structure of torsion-free cotorsion groups*

Definition. Let $Z(p)$ be the group of p-adic integers. The *height* of the p-adic integer π is the integer k ($\geqq 0$) such that $\pi \in p^k Z(p)$ but $\pi \notin p^{k+1} Z(p)$.

Lemma. Let $\sum_{\lambda \in \Lambda} C(p^\infty)_\lambda$ be a discrete direct sum of copies of $C(p^\infty)$, the quasi-cyclic group of type p (the index set Λ is arbitrary). Then $\operatorname{Hom}(C(p^\infty), \sum_{\lambda \in \Lambda} C(p^\infty)_\lambda)$ is isomorphic to a subgroup $\sum_\lambda{}' Z(p)_\lambda$ of the complete direct sum $\sum_\Lambda{}^{*} Z(p)_\lambda$, where, for each λ, $Z(p)_\lambda$ is a copy of the group of p-adic integers. In fact $\sum_{\lambda \in \Lambda}{}' Z(p)_\lambda$ is the set of those vectors $\langle \ldots, \pi_\lambda, \ldots \rangle$ which have only a finite number of components π_λ with height$=k$ for any integer $k = 0, 1, 2, \ldots$.

Proof. Let ε_λ be the projections of $\sum_\lambda C(p^\infty)_\lambda$ onto $C(p^\infty)_\lambda$. Then any $\varphi \in \operatorname{Hom}(C(p^\infty), \sum_\lambda C(p^\infty)_\lambda)$ uniquely determines the homomorphisms $\varphi\varepsilon_\lambda$ of $C(p^\infty)$ into $C(p^\infty)_\lambda$. Now $\varphi\varepsilon_\lambda$ defines a p-adic integer π_λ. So any $\varphi \in \operatorname{Hom}(C(p^\infty), \sum_\lambda C(p^\infty)_\lambda)$ uniquely determines a sequence of p-adic integers: $\langle \ldots, \pi_\lambda, \ldots \rangle$, such that π_λ is determined as $\varphi\varepsilon_\lambda$. Let $c_1, \ldots, c_n, \ldots$ be a generating system of $C(p^\infty)$ with $pc_1 = 0$, $pc_{n+1} = c_n$ ($n = 1, 2, \ldots$). Take an arbitrary generator c_k in $C(p^\infty)$. Then $(c_k)\varphi\varepsilon_\lambda = \langle \ldots, (c_k)\pi_\lambda, \ldots \rangle$ and we know that, in this vector, only a finite number of components are $\neq 0$. Now $\langle \ldots, (c_k)\pi_\lambda, \ldots \rangle \in \sum_\lambda C(p^\infty)_\lambda$ has only a finite number of components $\neq 0$ if and only if the number n_k of p-adic integers in this vector with height $< k$ is finite. Since this must hold for any integer k ($k = 0, 1, 2, \ldots$) it turns out that, for each k, the number of components π_λ with height k is finite.

Let φ define such a vector $\langle \ldots, \pi_\lambda, \ldots \rangle$ and assume that φ' defines the same vector $\langle \ldots, \pi_\lambda, \ldots \rangle$, then $\varphi - \varphi'$ maps each generator c_k upon 0, i.e. $\varphi = \varphi'$. If $\langle \ldots, \pi_\lambda, \ldots \rangle$ is an arbitrary vector with the mentioned properties, then $c_k \to \langle \dots, (c_k)\pi_\lambda, \dots \rangle$ induces a homomorphism φ of $C(p^\infty)$ into $\sum_\lambda C(p^\infty)_\lambda$ corresponding to $\langle \dots, \pi_\lambda, \dots \rangle$. This completes the proof of the lemma. It is now easy to derive the following

Theorem 1. Let D be a divisible torsion group and suppose

$$D = \sum_{p_i} \sum_{\alpha_{p_i}} C(p_i^\infty).$$

Then

$$\text{(1)} \qquad \mathrm{Hom}\,(Q/Z, D) \cong \sum_{p_j}{}^{*} \sum_{\alpha_{p_j}}{}' Z(p_j)$$

where the first (complete) sum is taken over all prime numbers p_j and, for each prime number p_j, the number of components π_λ with height $=k$ in $\langle \dots, \pi_\lambda, \dots \rangle \in \sum_{\alpha_{p_j}}' Z(p_j)$ is finite ($k=0, 1, 2, \dots$) (cf. Lemma).

Proof.

$$\mathrm{Hom}\,(Q/Z, D) = \mathrm{Hom}\,\Big(\sum_{p_j} C(p_j^\infty), \sum_{p_i} \sum_{\alpha_{p_i}} C(p_i^\infty)\Big) \cong \sum_{p_j}{}^{*} \mathrm{Hom}$$

$$\Big(C(p_j^\infty), \sum_{p_i} \sum_{\alpha_{p_i}} C(p_i^\infty)\Big) = \sum_{p_j}{}^{*} \mathrm{Hom}\,\Big(C(p_j^\infty), \sum_{\alpha_{p_j}} C(p_j^\infty)\Big) \cong \sum_{p_j}{}^{*} \sum_{\alpha_{p_j}}{}' Z(p_j) \text{ (Lemma).}$$

So all torsion-free cotorsion groups have the structure (1), or the torsion-free reduced algebraically compact groups have this form (cf. [2], p. 123 (j)). In particular any complete direct sum $\sum_{\alpha_p}^{*} Z(p)$ (p a fixed prime, α_p infinite cardinal) is such a torsion-free reduced algebraically compact group. We have $\mathrm{Hom}\,(C(p^\infty), \sum_{\alpha_p}^{*} C(p^\infty)) \cong \sum_{\alpha_p}^{*} \mathrm{Hom}\,(C(p^\infty), C(p^\infty)) \cong$ $\cong \sum_{\alpha_p}^{*} Z(p)$. But the torsion part of a direct product of α_p copies of each $C(p^\infty)$ is isomorphic to a direct sum of 2^{α_p} copies of $C(p^\infty)$, so $\mathrm{Hom}\,(C(p^\infty),$ $\sum_{\alpha_p}^{*} C(p^\infty)) = \mathrm{Hom}\,(C(p^\infty), \sum_{2^{\alpha_p}} C(p^\infty)) \cong \sum_{2^{\alpha_p}}' Z(p)$ by the lemma. We get that $\sum_{\alpha_p}^{*} Z(p) \cong \sum_{2^{\alpha_p}}' Z(p)$ for any infinite cardinal number α_p (p a fixed prime). The following remarks are due to Prof. L. Fuchs:

We have $\sum Z(p) \subset \sum' Z(p) \subset \sum^{*} Z(p)$, where $\sum'/\sum$ is the maximal divisible subgroup of $\sum^{*}/\sum$, the latter group being again algebraically compact. Actually $\sum'$ is the completion of $\sum$ in the n-adic or p-adic topology, (cf. [3], p. 379), so $\sum$ is dense in $\sum'$ which means that $\sum'/\sum$ is divisible ([3]. p. 380). Then $\sum^{*}$ will be the direct sum of $\sum'$ and a reduced algebraically compact group $\cong \sum^{*}/\sum'$.

2. *Direct sums of cotorsion groups*

Let $G = \sum_{\alpha \in I}^{*} G_\alpha$, where G_α is a cotorsion group for each α. It is known that G has a maximal torsion-free direct summand H, and $H=0$ if and only if G/G_t is divisible, (cf. [6], Theorem p. 242). The group H, as a direct summand of a cotorsion group, is cotorsion. Since H is torsion-free, it has the structure $\sum\limits_{p_j}^{*} \sum\limits_{\alpha_{p_j}}^{\prime} Z(p_j)$ according to the theorem 1. For some special cases the following property is useful.

Theorem 2. Let $G = \sum\limits_{\alpha \in I}^{*} G_\alpha$, where each G_α is a cotorsion group. Let $G_t = \sum\limits_{p} G_p$ be a direct decomposition of G_t into its primary components. Let B_p be a basic subgroup of G_p for each p and $B = \sum\limits_{p} B_p$. Then the maximal torsion-free direct summand H of G is 0 if and only if G/B is divisible.

Proof. $G_t/B = \sum\limits_{p} G_p / \sum\limits_{p} B_p \cong \sum\limits_{p} G_p/B_p$ is divisible, since G_p/B_p is divisible for each p. So G_t/B is a direct summand in G/B and $G/B \cong$ $\cong G_t/B \oplus G/G_t$. If G/B is divisible, then G/G_t is divisible and hence $H=0$ by the Theorem in [6]. Conversely, if $H=0$ then G/G_t is divisible and consequently G/B is divisible.

Now we consider some special cases of Theorem 2, which correspond to the cases discussed by E. WALKER in [6].

(i) Let $G = \sum\limits_{\alpha \in I}^{*} G_\alpha$, where each G_α is a torsion cotorsion group. Then each G_α is a torsion group of bounded order [6]. So G_α is a direct sum of cyclic groups (for each α). Let $G_\alpha^{(p)}$ be the p-primary component of G_α and G_p the p-primary component of G_t. Now $\sum\limits_{\alpha} G_\alpha^{(p)}$ is a basic subgroup of G_p for each p. And $B = \sum\limits_{p} (\sum\limits_{\alpha} G_\alpha^{(p)}) = \sum\limits_{\alpha} (\sum\limits_{p} G_\alpha^{(p)}) = \sum\limits_{\alpha} G_\alpha$, and G has a maximal torsion-free direct summand $H=0$ if and only if $\sum^{*} G_\alpha / \sum G_\alpha$ is divisible (Theorem 2). Now it is easy to see that $\sum^{*} G_\alpha / \sum G_\alpha$ is divisible if and only if $pG_\alpha = G_\alpha$ for almost all α. Hence the subgroup H of G is 0 if and only if, for any prime p, $p \nmid n_\alpha$ for almost all α, where n_α is the exponent of G_α for each α.

(ii) In the particular case that $\{G_\alpha\}_{\alpha \in I}$ is a set of cotorsion groups such that each G_α is of finite exponent n_α and such that $(n_\alpha, n_\beta) = 1$ if $\alpha \neq \beta$, it follows that the maximal torsion-free direct summand of $\sum\limits_{\alpha \in I}^{*} G_\alpha$ is 0 ([6], p. 243). If p is an arbitrary prime number and p/n_α for some n_α, then $p \nmid n_\beta$ for $\alpha \neq \beta$. Hence $pG_\beta = G_\beta$ for all $\beta \neq \alpha$. If p does not divide any n_α, then $pG_\alpha = G_\alpha$ for all α. In both cases one gets $p\, G_\alpha = G_\alpha$ for almost all α.

(iii) Let $G = \sum\limits_{i=1}^{\infty *} C(p^i)$, where $C(p^i)$ is a cyclic group of order p^i. Then $B = \sum\limits_{i=1}^{\infty} C(p^i)$ is a basic subgroup of G_t ([1], Theorem 29.6), and G/B is not divisible since $pC(p^i) \neq C(p^i)$ for all i. Hence G has a maximal torsion-free direct summand $H \neq 0$ and $H \cong \sum\limits_{p_j}^{*} \sum\limits_{\alpha_{p_j}}^{\prime} Z(p_j)$. Since G is p_k-divisible for any prime $p_k \neq p$, we must have that $\sum^*_{p_j} \sum'_{\alpha_{p_j}} Z(p_j)$ is p_k-divisible for any prime $p_k \neq p$, hence $H \cong \sum'_{\alpha_p} Z(p)$. It is our purpose to describe the structure of G/B in detail and we do this in the next section.

3. *The structure of* $\sum^{*\infty}_{i=1} C(p^i)/\sum^{\infty}_{i=1} C(p^i)$

First of all, it may be remarked that, by an analogous argument as in Lemma 4 of [5], it can be shown that Ext $(Q, \sum^*_{\lambda\in\Lambda} G_\lambda/\sum_{\lambda\in\Lambda} G_\lambda)=0$ (Λ is a countable set) for any set of groups G_λ. Hence $\sum^*_{\lambda\in\Lambda} G_\lambda/\sum_{\lambda\in\Lambda} G_\lambda$ is the direct sum of a divisible group and a cotorsion group ([3], § 2). Therefore $G/B = \sum^{*\infty}_{i=1} C(p^i)/\sum^{\infty}_{i=1} C(p^i)$ has the same structure and we know that the cotorsion part of it is not 0. As we have seen in the proof of Theorem 2, $G/B \cong G_t/B \oplus G/G_t$. Now G is a direct sum of a torsion-free cotorsion group H and an adjusted cotorsion group K. Since $G_t \subseteq K$, we get that $G/G_t \cong K/G_t \oplus H$. Now G_t is a p-group, containing B as a basic subgroup, hence G_t/B is divisible, in fact $G_t/B \cong \sum C(p^\infty)$. It is known that $|G_t/B| = 2^{\aleph_0}$ ([1], p. 102), so $G_t/B \cong \sum_{2^{\aleph_0}} C(p^\infty)$. Hence $G/B \cong \sum_{2^{\aleph_0}} C(p^\infty) \oplus \oplus K/G_t \oplus H$. Here $K/G_t \neq 0$, since $K/G_t = 0$ or $K = G_t$ would imply that G_t is a direct summand of $G = K \oplus H$, which is impossible, as is well known ([4], example 33, page 32). K/G_t is the maximal divisible torsion-free subgroup of G/B. Consider the set V of elements $\langle \ldots, a_i, \ldots\rangle + B \in G/B$ such that (i) the orders of the components a_i (in $C(p^i)$) are unbounded ($i = 1, 2, \ldots$) and (ii) there are only a finite number of components a_j with height $\leq k-1$ (in $C(p^j)$) for any integer $k \geq 1$. Then V is a divisible torsion-free subgroup of G/B, hence $V \subseteq K/G_t$. Since $|V| = 2^{\aleph_0}$, we get $|K/G_t| \geq 2^{\aleph_0}$. On the other hand, $K/G_t \subseteq G/B$, so $|K/G_t| \leq |G/B| = 2^{\aleph_0}$, so $|K/G_t| = 2^{\aleph_0}$. Hence $K/G_t \cong \sum_{2^{\aleph_0}} Q$, and $G/B \cong \sum_{2^{\aleph_0}} C(p^\infty) \oplus \sum_{2^{\aleph_0}} Q \oplus H$. We know that $H \cong \sum'_{\alpha_p} Z(p)$ and $G/B/p(G/B) \cong H/pH$. So we have to compute the rank of the p-group $G/B/p(G/B)$. Since B is pure in G_t we get $pB = B \cap pG_t$ which implies that $pB = B \cap pG$. Now $pB = B \cap pG$ implies $B/pB = B/B \cap pG \cong B + pG/pG$, hence $G/pG/B/pB \cong G/B + pG \cong \cong G/B/pG + B/B = G/B/p(G/B)$. Then $G/pG \cong \sum^*_{\aleph_0} C(p)$ and $B/pB \cong \cong \sum_{\aleph_0} C(p)$ imply that the rank of $G/B/p(G/B) = 2^{\aleph_0}$. Consequently the rank of $H/pH = 2^{\aleph_0}$.

By Prop. 2.1. [3], p. 371, there exists essentially one torsion-free cotorsion group H with the dimension of H/pH (as a vector space over the prime field of characteristic p) equal to $2^{\aleph_0}$, in fact, $H = \mathrm{Hom}\,(Q/Z, \sum_{2^{\aleph_0}} C(p^\infty))$. By Theorem 1, we get that $H \cong \sum'_{2^{\aleph_0}} Z(p)$. So we have shown:

Theorem 3. Let $G = \sum^{*\,\infty}_{i=1} C(p^i)$ and $B = \sum^{\infty}_{i=1} C(p^i)$, where $C(p^i)$ is a cyclic group of order p^i (p a fixed prime). Then the maximal torsion-free direct summand H of G has the structure $\sum'_{2^{\aleph_0}} Z(p)$ or $\sum^*_{\aleph_0} Z(p)$ (see the remarks after Theorem 1). Moreover

$$G/B \cong \sum_{2^{\aleph_0}} Q \oplus \sum_{2^{\aleph_0}} C(p^\infty) \oplus \sum'_{2^{\aleph_0}} Z(p).$$

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